Introduction to Inertial Navigation and Kalman filtering

INS Tutorial,
Norwegian Space Centre
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Kenneth Gade, FFI
Outline

• Notation
• Inertial navigation
• Aided inertial navigation system (AINS)
• Implementing AINS
• Initial alignment (gyrocompassing)
• AINS demonstration
Kinematics

• Mathematical model of physical world using
  – *Point*, represents a *position*/particle (affine space)
  – *Vector*, represents a *direction* and *magnitude* (vector space)
Coordinate frame

- One point (representing position)
- Three basis vectors (representing orientation)

→ 6 degrees of freedom
→ Can represent a rigid body
## Important coordinate frames

<table>
<thead>
<tr>
<th>Frame symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>Inertial</td>
</tr>
<tr>
<td>$E$</td>
<td>Earth-fixed</td>
</tr>
<tr>
<td>$B$</td>
<td>Body-fixed</td>
</tr>
<tr>
<td>$N$</td>
<td>North-East-Down (local level)</td>
</tr>
<tr>
<td>$L$</td>
<td>Local level, wander azimuth (as $N$, but not north-aligned =&gt; nonsingular)</td>
</tr>
</tbody>
</table>
**Local level frames**

<table>
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<th>Frame symbol</th>
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</thead>
<tbody>
<tr>
<td>( N )</td>
<td>North-East-Down (local level)</td>
</tr>
<tr>
<td>( L )</td>
<td>Local level, wander azimuth (as ( N ), but not north-aligned =&gt; nonsingular)</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
R_{EL} & \iff \text{longitude, latitude, wander azimuth} \\
R_{NB} \quad R_{LB} & \iff \text{roll, pitch, yaw}
\end{align*}
\]
General vector notation

*Coordinate free* vector (suited for expressions/deductions): \( \vec{x} \)

Sum of components along the basis vectors of \( E \) (\( \vec{b}_{E,i}, \vec{b}_{E,j}, \vec{b}_{E,k} \)):

\[
\vec{x} = x_i \vec{b}_{E,i} + x_j \vec{b}_{E,j} + x_k \vec{b}_{E,k}
\]

Vector *decomposed in frame* \( E \) (suited for computer implementation):

\[
\vec{x}^E = \begin{bmatrix} x_i \\ x_j \\ x_k \end{bmatrix}
\]
# Notation for position, velocity, acceleration

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{p}_{AB}$</td>
<td>$B - A$</td>
<td><strong>Position vector.</strong> A vector whose length and direction is such that it goes from the origin of $A$ to the origin of $B$.</td>
</tr>
<tr>
<td>$C \vec{v}_{AB}$</td>
<td>$C \frac{d}{dt}(\vec{p}_{AB})$</td>
<td><strong>Generalized velocity.</strong> Derivative of $\vec{p}_{AB}$, relative to coordinate frame $C$.</td>
</tr>
<tr>
<td>$\vec{v}_{AB}$</td>
<td>$A \vec{v}_{AB}$</td>
<td><strong>Standard velocity.</strong> The velocity of the origin of coordinate frame $B$ relative to coordinate frame $A$. (The frame of observation is the same as the origin of the differentiated position vector.) Note that the underline shows that both orientation and position of $A$ matters (whereas only the position of $B$ matters)</td>
</tr>
<tr>
<td>$C \vec{a}_{AB}$</td>
<td>$C \frac{d^2}{(dt)^2}(\vec{p}_{AB})$</td>
<td><strong>Generalized acceleration.</strong> Double derivative of $\vec{p}_{AB}$, relative to coordinate frame $C$.</td>
</tr>
<tr>
<td>$\vec{a}_{AB}$</td>
<td>$A \vec{a}_{AB}$</td>
<td><strong>Standard acceleration.</strong> The acceleration of the origin of coordinate frame $B$ relative to coordinate frame $A$.</td>
</tr>
</tbody>
</table>
### Notation for orientation and angular velocity

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\theta}_{AB} )</td>
<td>( \vec{k}<em>{AB} \cdot \beta</em>{AB} )</td>
<td><strong>Angle-axis product.</strong> ( \vec{k}<em>{AB} ) is the axis of rotation and ( \beta</em>{AB} ) is the angle rotated.</td>
</tr>
<tr>
<td>( R_{AB} )</td>
<td>(to be published)</td>
<td><strong>Standard rotation matrix.</strong> Mostly used to store orientation and decompose vectors in different frames, ( \vec{x}^A = R_{AB} \vec{x}^B ). Notice the “rule of closest frames”.</td>
</tr>
<tr>
<td>( \vec{\omega}_{AB} )</td>
<td>(to be published)</td>
<td><strong>Angular velocity.</strong> The angular velocity of coordinate frame ( B ), relative to coordinate frame ( A ).</td>
</tr>
</tbody>
</table>
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• Notation
• **Inertial navigation**
• Aided inertial navigation system (AINS)
• Implementing AINS
• Initial alignment (gyrocompassing)
• AINS demonstration
Navigation

Navigation:
Estimate the position, orientation and velocity of a vehicle

Inertial navigation:

_Inertial sensors_ are utilized for the navigation
Inertial Sensors

Based on inertial principles, *acceleration* and *angular velocity* are measured.

- Always relative to *inertial space*
- Most common inertial sensors:
  - *Accelerometers*
  - *Gyros*
Accelerometers (1:2)

By attaching a mass to a spring, measuring its deflection, we get a simple accelerometer.

– To improve the dynamical interval and linearity and also reduce hysteresis, a control loop, keeping the mass close to its nominal position can be applied.
Accelerometers (2:2)

- **Gravitation** is also measured (Einstein's principle of equivalence)

- Total measurement called **specific force**, \( \vec{f}_{IB} = \vec{a}_{IB} - \vec{g}_B = \vec{a}_{IB} - \frac{\vec{F}_{\text{gravitation}}}{m} \)

- Using 3 (or more) accelerometers we can form a 3D specific force measurement:

\[
\vec{f}_{IB}^B
\]

This means: Specific force of the body system (\( B \)) relative inertial space (\( I \)), decomposed in the body system.

Good commercial accelerometers have an accuracy in the order of 50 \( \mu \text{g} \).
Gyros (1:3)

Gyros measure angular velocity relative inertial space: $\hat{\omega}_{IB}$

Principles:

- **Maintain angular momentum** *(mechanical gyro).* A spinning wheel will resist any change in its angular momentum vector relative to inertial space. Isolating the wheel from vehicle angular movements by means of gimbals and then output the gimbal positions is the idea of a mechanical gyro.

Figure: Caplex (2000)
Gyros (2:3)

- **The Sagnac-effect.** The inertial characteristics of light can also be utilized, by letting two beams of light travel in a loop in opposite directions. If the loop rotates clockwise, the clockwise beam must travel a longer distance before finishing the loop. The opposite is true for the counter-clockwise beam. Combining the two rays in a detector, an interference pattern is formed, which will depend on the angular velocity.

The loop can be implemented with 3 or 4 mirrors (*Ring Laser Gyro*), or with optical fibers (*Fiber Optic Gyro*).
Gyros (3:3)

- **The Coriolis-effect.** Assume a mass that is vibrating in the radial direction of a rotating system. Due to the Coriolis force working perpendicular to the original vibrating direction, a new vibration will take place in this direction. The amplitude of this new vibration is a function of the angular velocity. MEMS gyros (MicroElectroMechanical Systems), “tuning fork” and “wineglass” gyros are utilizing this principle. Coriolis-based gyros are typically cheaper and less accurate than mechanical, ring laser or fiber optic gyros.

Figure: Titterton & Weston (1997)
Several inertial sensors are often assembled to form an *Inertial Measurement Unit (IMU)*.

- Typically the unit has 3 accelerometers and 3 gyros (x, y and z).

In a *strapdown IMU*, all inertial sensors are rigidly attached to the unit (no mechanical movement).

In a *gimballed IMU*, the gyros and accelerometers are isolated from vehicle angular movements by means of gimbals.
Example (Strapdown IMU)

Honeywell HG1700 ("medium quality"):

- 3 accelerometers, accuracy: 1 mg
- 3 ring laser gyros, accuracy: 1 deg/h
- Rate of all 6 measurements: 100 Hz
Inertial Navigation

An IMU (giving $\mathbf{f}_{IB}^B$ and $\omega_{IB}^B$) is sufficient to navigate relative to inertial space (no gravitation present), given initial values of velocity, position and orientation:

- Integrating the sensed acceleration will give velocity.
- A second integration gives position.
- To integrate in the correct direction, orientation is needed. This is obtained by integrating the sensed angular velocity.
Terrestrial Navigation

In *terrestrial navigation* we want to navigate relative to the Earth \((E)\). Since earth is not an inertial system, and gravity is present, the inertial navigation becomes somewhat more complex:

- Earth angular rate must be compensated for in the gyro measurements:
  \[
  \mathbf{\omega}^B_{EB} = \mathbf{\omega}^B_{IB} - \mathbf{\omega}^B_{IE}
  \]

- Accelerometer measurement compensations:
  - Gravitation
  - Centrifugal force (due to rotating Earth)
  - Coriolis force (due to *movement* in a rotating frame)
Navigation Equations

Strapdown IMU, wander azimuth Local system (L), spherical earth. Not included: vertical direction, gravity calculation.
Inertial Navigation System (INS)

The combination of an IMU and a computer running navigation equations is called an *Inertial Navigation System (INS)*.

Due to errors in the gyros and accelerometers, an INS will have **unlimited drift** in velocity, position and attitude.

The quality of an IMU is often expressed by expected position drift per hour (1σ). Examples (classes):

- HG1700 is a **10 nautical miles per hour IMU**.
- HG9900 is a **1 nautical mile per hour IMU**.
# Categorization: IMU technology and IMU performance

<table>
<thead>
<tr>
<th>Class</th>
<th>Position performance</th>
<th>Gyro technology</th>
<th>Accelerometer technology</th>
<th>Gyro bias</th>
<th>Acc bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Military grade&quot;</td>
<td>1 nmi / 24 h</td>
<td>ESG, RLG, FOG</td>
<td>Servo accelerometer</td>
<td>&lt; 0.005°/h</td>
<td>&lt; 30 µg</td>
</tr>
<tr>
<td>Navigation grade</td>
<td>1 nmi / h</td>
<td>RLG, FOG</td>
<td>Servo accelerometer, Vibrating beam</td>
<td>0.01°/h</td>
<td>50 µg</td>
</tr>
<tr>
<td>Tactical grade</td>
<td>&gt; 10 nmi / h</td>
<td>RLG, FOG</td>
<td>Servo accelerometer, Vibrating beam, MEMS</td>
<td>1°/h</td>
<td>1 mg</td>
</tr>
<tr>
<td>AHRS</td>
<td>NA</td>
<td>MEMS, RLG, FOG, Coriolis</td>
<td>MEMS</td>
<td>1 - 10°/h</td>
<td>1 mg</td>
</tr>
<tr>
<td>Control system</td>
<td>NA</td>
<td>Coriolis</td>
<td>MEMS</td>
<td>10 - 1000°/h</td>
<td>10 mg</td>
</tr>
</tbody>
</table>
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Aided inertial navigation system

To limit the drift, an INS is usually aided by other sensors that provide direct measurements of the integrated quantities.

Examples of aiding sensors:

<table>
<thead>
<tr>
<th>Sensor:</th>
<th>Measurement:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure meter</td>
<td>Depth/height</td>
</tr>
<tr>
<td>Magnetic compass</td>
<td>Heading</td>
</tr>
<tr>
<td>Doppler velocity log</td>
<td>$\mathbf{v}<em>{EB}^B$ (or $\mathbf{v}</em>{WB}^B$, water)</td>
</tr>
<tr>
<td>Underwater transponders</td>
<td>Range from known position</td>
</tr>
<tr>
<td>GPS</td>
<td>$\mathbf{p}_{EB}^E$</td>
</tr>
<tr>
<td>GPS (Doppler shift)</td>
<td>$\mathbf{v}_{EB}^E$</td>
</tr>
<tr>
<td>Multi-antenna GPS</td>
<td>Orientation</td>
</tr>
</tbody>
</table>
Sensor error models

Typical error models for IMU, Doppler velocity log and others:

- white noise
- colored noise (1st order Markov)
- scale factor error (constant)
- misalignment error (constant)
Kalman Filter

A Kalman filter is a recursive algorithm for estimating states in a system.

Examples of states:
- Position, velocity etc for a vehicle
- pH-value, temperature etc for a chemical process

Two sorts of information are utilized:

• **Measurements** from relevant sensors
• A **mathematical model** of the system (describing how the different states depend on each other, and how the measurements depend on the states)

In addition the *accuracy* of the measurements and the model must be specified.
Kalman Filter Algorithm

Description of the recursive Kalman filter algorithm, starting at $t_0$:

1. At $t_0$, the Kalman filter is provided with an *initial estimate*, including its uncertainty (covariance matrix).

2. Based on the mathematical model and the initial estimate, a new estimate valid at $t_1$ is *predicted*. The uncertainty of the *predicted estimate* is calculated based on the initial uncertainty, and the accuracy of the model (*process noise*).

3. Measurements valid at $t_1$ give new information about the states. Based on the accuracy of the measurements (*measurement noise*) and the uncertainty in the predicted estimate, the two sources of information are weighed and a new *updated estimate* valid at $t_1$ is calculated. The uncertainty of this estimate is also calculated.

4. At $t_2$, a new estimate is predicted as in step 2, but now based on the updated estimate from $t_1$.

... The prediction and the following update are repeated each time a new measurement arrives.

*If the models/assumptions are correct, the Kalman filter will deliver optimal estimates.*
Kalman Filter Design for Navigation

**Objective:** Find the vehicle position, attitude and velocity with the best accuracy possible

**Possible basis:**
- Sensor measurements \((\text{measurements})\)
- System knowledge \((\text{mathematical model})\)
- Control variables \((\text{measurements})\)

We utilize sensor measurements and knowledge of their behavior (error models).

This information is combined by means of an error-state Kalman filter.
Example: HUGIN

**DGPS**: Differential Global Positioning System

**HiPAP**: High Precision Acoustic Positioning

**DVL**: Doppler Velocity Log
Measurements

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Measurement</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMU</td>
<td>Angular velocity, specific force</td>
<td>$\omega_{IB}^B$, $f_{IB}^B$</td>
</tr>
<tr>
<td>DGPS/HiPAP</td>
<td>Horizontal position measurement</td>
<td>$p_{EB}^E$</td>
</tr>
<tr>
<td>Pressure sensor</td>
<td>Depth</td>
<td></td>
</tr>
<tr>
<td>DVL</td>
<td>AUV velocity (relative the seabed) projected into the body (B) coordinate system</td>
<td>$v_{EB}^B$</td>
</tr>
<tr>
<td>Compass</td>
<td>Heading (relative north)</td>
<td>$\psi_{north}$</td>
</tr>
</tbody>
</table>

To make measurements for the error-state Kalman filter we form differences of all redundant information. This can be done by running navigation equations on the IMU-data, and compare the outputs with the corresponding aiding sensors.

The INS and the aiding sensors have complementary characteristics.
Based on the measurements and sensor error models, the Kalman filter estimates errors in the navigation equations and all colored sensor errors.
Optimal Smoothing

**Smoothed estimate:** Optimal estimate based on all logged measurements (from both history and future)

Smoothing gives:
- Improved accuracy (number of relevant measurements doubled)
- Improved robustness
- Improved integrity
- Estimate in accordance with process model

First the ordinary Kalman filter is run through the entire time series, saving all estimates and covariance matrices. The saved data is then processed recursively backwards in time using an optimal *smoothing algorithm* adjusting the filtered estimates (Rauch-Tung-Striebel implementation).
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Practical navigation processing

Any vehicle with an IMU and some aiding sensors, can use the AINS to find its position, orientation and velocity.

Typical implementation:

- **Real-time navigation**
- **Post-processed navigation**
NavLab

NavLab (Navigation Laboratory) is one common tool for solving a variety of navigation tasks.

Development started in 1998

Main focus during development:
- Solid theoretical foundation (competitive edge)

Structure:

Simulator (can be replaced by real measurements)  Estimator (can interface with simulated or real measurements)

- IMU Simulator
- Position measurement Simulator
- Depth measurement Simulator
- Velocity measurement Simulator
- Compass Simulator
- Navigation Equations
- Error state Kalman filter
- Make Kalman filter measurements (differences)
- Optimal Smoothing
- Filtered estimates and covariance matrices
- Smoothed estimates and covariance matrices
Simulator

- Trajectory simulator
  - Can simulate any trajectory in the vicinity of Earth
  - No singularities

- Sensor simulators
  - Most common sensors with their characteristic errors are simulated
  - All parameters can change with time
  - Rate can change with time

Figure: NavLab
NavLab Usage

Main usage:
- Navigation system research and development
- Analysis of navigation system
- Decision basis for sensor purchase and mission planning
- Post-processing of real navigation data
- Sensor evaluation
- Tuning of navigation system and sensor calibration

Users:
- Research groups (e.g. FFI (several groups), NATO Undersea Research Centre, QinetiQ, Kongsberg Maritime, Norsk Elektro Optikk)
- Universities (e.g. NTNU, UniK)
- Commercial companies (e.g. C&C Technologies, Geoconsult, FUGRO, Thales Geosolutions, Artec Subsea, Century Subsea)
- Norwegian Navy

Vehicles navigated with NavLab: AUVs, ROVs, ships and aircraft

For more details, see www.navlab.net
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Initial alignment (gyrocompassing)

Basic problem:
Find the orientation of a vehicle \((B)\) relative to Earth \((E)\) by means of an IMU and additional knowledge/measurements.

Note: An optimally designed AINS inherently gyrocompasses optimally. However, a starting point must be within tens of degrees due to linearizations in the Kalman filter => gyrocompassing/initial alignment is treated as a separate problem.

Solution: Find Earth-fixed vectors decomposed in \(B\). One vector gives two degrees of freedom in orientation.

Relevant vectors:
- Gravity vector
- Angular velocity of Earth relative to inertial space, \(\vec{\omega}_{IE}\)
Finding the vertical direction (roll and pitch)

Static condition: Accelerometers measure gravity, thus roll and pitch are easily found.

Dynamic condition: The acceleration component of the specific force measurement must be found (\( f_{IB}^B = a_{IB}^B - g_B^B \))

=> additional knowledge is needed

The following can give acceleration knowledge:
- External position measurements
- External velocity measurements
- Vehicle model
Finding orientation about the vertical axis: Gyrocompassing

**Gyrocompassing:** The concept of finding orientation about the vertical axis (yaw/heading) by measuring the direction of Earth's axis of rotation relative to inertial space $\mathbf{w}_{IE}$

- Earth rotation is measured by means of gyros
Gyrocompassing under static condition

Static condition \((\vec{\omega}_{EB} = 0)\):
A gyro triad fixed to Earth will measure the 3D direction of Earth’s rotation axis \((\vec{\omega}^B_{IB} = \vec{\omega}^B_{IE})\)

- To find the yaw-angle, the down-direction (vertical axis) found from the accelerometers is used.
- Yaw will be less accurate when getting closer to the poles, since the horizontal component of \(\vec{\omega}_{IE}\) decreases \((1/\cos(\text{latitude}))\). At the poles \(\vec{\omega}_{IE}\) is parallel with the gravity vector and no gyrocompassing can be done.

Figure assumes \(x\)- and \(y\)-gyros in the horizontal plane:
Gyrocompassing under dynamic conditions (1:2)

Dynamic condition:

- Gyros measure Earth rotation + vehicle rotation, \( \mathbf{\omega}_{IB}^B = \mathbf{\omega}_{IE}^B + \mathbf{\omega}_{EB}^B \)
- Challenging to find \( \mathbf{\omega}_{IE}^B \) since \( \mathbf{\omega}_{EB}^B \) typically is several orders of magnitude larger
Gyrocompassing under dynamic conditions (2:2)

Under dynamic conditions gyrocompassing can be performed if we know the direction of the gravity vector over time relative to inertial space.

– The gravity vector will rotate about Earth's axis of rotation:

The change in gravity direction due to own movement over the curved Earth can be compensated for if the velocity is known (4 m/s north/south => 1° error at lat 60°)
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AINS demonstration - simulation

Position vs time

Lat [deg]

Time [s]

43.994
43.996
43.998
44

0
200
400
600
800
1000

0
200
400
600
800
1000

10
10.002
10.004
10.006
10.008

0
200
400
600
800
1000

-751
-749
-748
-750
-751

0
200
400
600
800
1000

Figures: NavLab

Timing overview

True trajectory
Measurement
Calculated value from navigation equations
Estimate from real-time Kalman filter
Smoothed estimate

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Position in meters ($p_{MB}^M$) vs time

- True trajectory
- Measurement
- Calculated value from navigation equations
- Estimate from real-time Kalman filter
- Smoothed estimate

Posm white (1σ): 3 m
Posm bias (1σ): 4 m
$T_{bias}$: 60 s
Posm rate: 1/60 Hz

Figure: NavLab
Position estimation error

Est error in naveq position and \( \text{std} (\delta n_{naveq, x+y}^L + \delta z_{naveq}) \)

Figure: NavLab
Attitude

Figure: NavLab
Attitude estimation error

Est error in naveq attitude and \( \text{std (} e_{LB,naveq}^{L} \text{)} \)

- \( x \) (deg):
  - \( x \times 10^{-3} \)
  - std = 0.010885
  - std = 0.00041765

- \( y \) (deg):
  - \( y \times 10^{-3} \)
  - std = 0.010998
  - std = 0.00039172

- \( z \) (deg):
  - \( z \times 10^{-3} \)
  - std = 0.42831
  - std = 0.022396

Figure: NavLab
AINS demonstration - real data

- Data from Gulf of Mexico
- Recorded with HUGIN 3000
Position (real data)

Longitude vs latitude

Long [deg]  Lat [deg]

-90.304  28.134
-90.302  28.132
-90.3  28.13
-90.298  28.128
-90.296  28.126
-90.294  28.124
-90.292  28.122

True trajectory
Measurement
Calculated value from navigation equations
Estimate from real-time Kalman filter
Smoothed estimate

Figure: NavLab
USBL wildpoint (outlier)

2D trajectory in meters, $p_{MB}^M$

- True trajectory
- Measurement
- Calculated value from navigation equations
- Estimate from real-time Kalman filter
- Smoothed estimate

Figure: NavLab
Verification of Estimator Performance

Verified using various simulations

Verified by mapping the same object repeatedly

HUGIN 3000 @ 1300 m depth:

- Std North = 1.17 m
- Std East = 1.71 m
Navigating aircraft with NavLab

- Cessna 172, 650 m height, much turbulence
- Simple GPS and IMU (no IMU spec. available)

Line imager data

Positioned with NavLab (abs. accuracy: ca 1 m verified)
Conclusions

- An **aided inertial navigation system** gives:
  - optimal solution based on all available sensors
  - all the relevant data with high rate
- If real-time data not required, **smoothing** should always be used to get maximum accuracy, robustness and integrity
Extra slides
Typical sensor/method for the 6 DOF and velocity

**Horizontal position:**
- Range from known positions (GPS, underwater transponders, etc)
- Terrain navigation

**Vertical position:**
- Pressure sensor
- Range from known positions (GPS, underwater transponders)

**Velocity:**
- Acoustic Doppler velocity log (DVL)
- GPS Doppler shift

**Heading:**
- Magnetic compass
- Gyrocompassing
- DVL+ position measurements (velocity required)
- IMU + position/velocity in E (acceleration required)
- Multi-antenna GPS

**Roll, pitch:**
- IMU + g-vector
- Multi-antenna GPS
The different outputs

- **Measurement (from aiding sensor)**
  - low rate
  - high frequency errors
  - stable

- **Navigation Equations**
  - high rate
  - very good at high frequencies
  - unlimited drift

- **Real time Kalman filter**
  - desired rate
  - small jumps due to unexpected measurements

- **Smoothed estimate**
  - desired rate
  - in accordance with process model